Exact Solution of the Simple Pendulum

Jacobi Elliptic Functions

Noah Roetman

Equation of Motion

$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$

Mechanical Energy

$$\boxed{E(t) = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta)}$$

Conservation of mechanical energy:

$$E(\theta) = E(0)$$

$$\frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta) = mgl(1-\cos\theta_0)$$

Thus, simplifying by ml^2:

$$\frac{1}{2}\dot{\theta}^2 + \frac{g}{l}(1-\cos\theta) = \frac{g}{l}(1-\cos\theta_0)$$

We know that:

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

$$\dot{\theta}^2 = \frac{4g}{l}\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right)$$

Change of Variables

Let:

$$k = \sin\frac{\theta_0}{2}, \quad \sin\frac{\theta}{2} = k\sin\varphi$$

Then:

$$\cos^2\frac{\theta}{2} = 1 - k^2\sin^2\varphi$$

Differentiating:

$$\frac{1}{2}\cos\frac{\theta}{2}\,d\theta = k\cos\varphi\,d\varphi$$

So:

$$d\theta = \frac{2k\cos\varphi}{\sqrt{1-k^2\sin^2\varphi}}\,d\varphi$$

And:

$$\sqrt{k^2-\sin^2\frac{\theta}{2}} = k\cos\varphi$$

Elliptic Integral

Substituting into the differential equation:

$$dt = \sqrt{\frac{l}{g}}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

Integrating:

$$t = \sqrt{\frac{l}{g}}\int_0^{\varphi}\frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

This is the incomplete elliptic integral of the first kind:

$$F(\varphi, k)$$

Solution with Jacobi Functions

Using the Jacobi elliptic functions am and sn:

$$\varphi = \mathrm{am}\Big(\sqrt{\frac{g}{l}}t, k\Big), \quad \sin\varphi = \mathrm{sn}\Big(\sqrt{\frac{g}{l}}t, k\Big)$$

So:

$$\sin\frac{\theta}{2} = k\sin\varphi$$

Therefore:

$$\theta(t) = 2\arcsin\left(k\,\mathrm{sn}\Big(\sqrt{\frac{g}{l}}t, k\Big)\right)$$

From which we get the final solution:

$$\boxed{\theta(t) = 2\arcsin\left(\sin\frac{\theta_0}{2}\cdot\mathrm{sn}\Big(\sqrt{\frac{g}{l}}t, \sin\frac{\theta_0}{2}\Big)\right)}$$